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(September 13, 2001)

The muon anomalous g value, $a_\mu = (g-2)/2$, is calculated up to one-loop level in noncommutative QED. We argue that relativistic muon in E821 experiment nearly always stays at the lowest Landau level. So that spatial coordinates of muon do not commute each other. Using parameters of E821 experiment, $B = 14.5$KG and muon energy 3.09GeV/c, we obtain the noncommutativity correction to a_μ is about 1.57×10^{-9}, which significantly make standard model prediction close to experiment.

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I. INTRODUCTION

Recently, noncommutative field theory becomes an active subject again maybe because of development of string/M theory, in which noncommutative Yang-Mills theory arise in a definite limit of string theory with a nonzero background field [1,2]. In this sense noncommutativity is motivated by some profound but beyond the current experiment physics thoughts, such as the long-held belief that space-time must change its nature at distances comparable to the Planck scale. The idea of noncommutativity of spacetime cooridnates, however, is quite old [3] in physics and mathematics. It also relates to some today-obversable physics. The classic example (though not always discussed using the language of noncommutative field theory) is the theory of electrons in a magnetic field projected to the lowest Landau level, which is naturally thought of as a noncommutative field theory. So far, the theoretical studies on noncommutative field theory have achieved a number of results [4]. It was shown that noncommutative quantun field theory exhibits an interesting UV-IR divergence mixing [5], and noncommutative gauge theory is renormalizable and gauge invariant at one-loop level at least [6]. In this letter, in terms of these theoretical studies, we will study a rather phenomenological issue: the muon anomalous magnetic moment in noncommutative QED.

The anomalous magnetic moments (AMM's) of electron and muon have been taken as one of the most precise and beautiful tests of the validity of quantum field theories like QED and the standard model (SM). They have been calculated and measured to extremely high precison, and have always been shown a good agreement each other. This situation, however, seems to has been changed recently due to E821 experiment at BNL measuing the anomalous magnetic moment of muon to a precision of 1.3 parts per million (ppm), which gave a deviation from the SM theoretical value

$$\delta a_\mu = a_\mu(\text{exp}) - a_\mu(\text{SM}) = 43(16) \times 10^{-10}. \tag{1}$$

It is 2.6 times the normal deivation [7]. This result has been treated as an indication of new physics and caused extensive interest in many recent literatures [8]. Meanwhile, the more careful theorectical study in framework of SM is still going on [9] for confirming SM prediction. Besides of these considerations, we should consider the effect of environment of measure. Sometimes the environment of measure in experiment not only enters the systematic error, but also changes the physics. A simplest example is also for an electron motioning in a (homogeneous, for simple) magnetic field. When electron stays at the lowest Landau level, position coordinates of electron which are perpendicular to the magnetic field $\mathbf{B}$ do not commute each other (for simple, we assume $B_1 = B_2 = 0$, $B_3 = B$=constant)

$$[x^i, x^j] = i\epsilon^{ij}\frac{\hbar c}{eB}, \qquad i, j = 1, 2. \tag{2}$$

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It means that electrons in the lowest Landau level should be described by a nonlocal field theory rather than usual local quantum field theory like QED. In usual experiment environment, electron (or muon) is easily excitated to higher Landau level, and this effect is covered. But when there is large probability that electron (or muon) stays at the lowest Landau level, the above effect has to be considered in theoretical prediction.

Now let us focus on the measure of muon AMM in E821 experiment. There are two main characteristices in E821 experiment: a homogeneous magnetic field of 14.5KG and highly relativistic muon with energy 3.09Gev/c ($\gamma_{\rm L} \simeq 29.3$) which dilates lifetime of muon to $\gamma_{\rm L}\tau \simeq 64.4\mu$s. However, there is effect of synchronous radiation for circumnutation of highly relativistic muon in magnetic field. It forces that muon loses energy ceaselessly and nearly always stays at the lowest Landau level. In this sense, muon physics in this situation should be described by a nonlocal quantum field theory (in particular, noncommutative QED) instead of usual QED. In this letter, therefore, noncommutative QED will be used to calculate muon AMM in E821 experiment.

The letter is organized as follows. In sect. 2, the basic notation of noncommutative field theory is reviewed. In sect. 3, we calculate the muon AMM in noncommutative QED to one-loop, and correction on muon AMM due to noncommutative coordinates is obtained. In sect. 4, we first evaluate the numer result of the noncommutative correction. Then we devote a brief summary.

II. NONCOMMUTATIVE QUANTUM ELECTRODYNAMICS IN $\mathcal{M}^4$

Consider four dimension space-time with coordinates $x^\mu,\ \mu = 0, ..., 3$ which obey the following commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{3}$$

where $\theta^{\mu\nu}$ is a constant asymmetric tensor. In particular, in this letter we focus on only spatial coordinates are noncommutative, i.e., $\theta^{0i} = 0,\ \theta^{ij} \neq 0,\ i, j = 1, 2, 3$. By noncommutative space-time one means the algebra $\mathcal{A}_\theta$ generated by the x^μ satisfying (3), together with some extra conditions on the allows expressions of the x^μ. The elements of $\mathcal{A}_\theta$ can be identified with ordinary functions on $\mathcal{M}^4$, with the product of two functions f and g given by the Moyal formula (or star product):

$$(f \star g)(x) = \exp[\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}]f(x_1)g(x_2)|_{x_1=x_2=x} \tag{4}$$

A field theory is defined as usual by constructing an action, but replace ordinary product by star product. For example, the action of noncommutative quantum electrodynaimcs (NCQED) is

$$S = \int d^4x\{\bar\psi \star (i\partial\!\!\!/ - m) \star \psi + e\bar\psi \star A\!\!\!/ \star \psi - \frac{1}{4}F^{\mu\nu} \star F_{\mu\nu}\}. \tag{5}$$

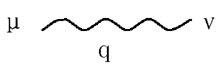
This action exhibits an U(1) gauge symmetry, and the guage field is provided by a real vector function, $A_\mu(x)$, on $\mathcal{M}^4$. But field strength $F_{\mu\nu}$ for this guage field now reads $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_\star$, where $[A_\mu, A_\nu]_\star = (A_\mu \star A_\nu)(x) - (A_\nu \star A_\mu)(x)$. Thus U(1) noncommutative QED is similar to usual Yang-Mills theory.

It has been shown that the basic structure of renomalization of noncummutative field theory is rather different from usual commutative guage theory. In other words, the UV properties are controlled by the planar diagrams, while nonplanar diagrams generally lead through what is called "UV/IR mixing" [5] to new IR phenomena. The limit $\theta \to 0$ in these theories is non-analytic. Meanwhile, pure U(N) noncommutative guage theory is renormalizable and gauge invariant, at least at one-loop level: U(1) case is studied in ref. [6], and general U(N) case is studied in refs. [10–13]. In addition, it has been also shown that quantum noncommutative field theory is unitary theory if it defines in Euclidean space or noncommutatiivity is purely spatial ($\theta_{0i} = 0$) in Minkowski space-time. Therefore, noncommutative QED considering by this letter is well-defined even at quantum level although it is not Lorentz covariant (The noncovairance is easily understood since the background field is anisotropic).

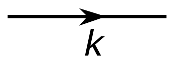
Using basic properties of star product:

$$\begin{gathered}\int d^4x(f \star g)(x) = \int d^4x(g \star f)(x) \\ \int d^4x(f \star g \star h)(x) = \int d^4x(h \star f \star g)(x) \\ (f \star g) \star h(x) = f \star (g \star h)(x)\end{gathered} \tag{6}$$

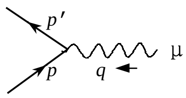
we can obtain the following Feynman rules for NCQED (Feynman-t' Hooft gauge)



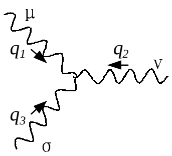
$$\frac{-ig_{\mu\nu}}{q^2+i\epsilon}$$



$$\frac{i}{\not{k}-m+i\epsilon}$$



$$ie\gamma^\mu e^{-\frac{i}{2}p\times_\theta p'}$$



$$ie(e^{-\frac{i}{2}q_2\times_\theta q_3}-e^{\frac{i}{2}q_2\times_\theta q_3})\{g_{\mu\sigma}(q_1-q_3)_\nu$$
$$+g_{\mu\nu}(q_2-q_1)_\sigma+g_{\nu\sigma}(q_3-q_2)_\mu\}$$

where $p\times_\theta k=p_\mu\theta^{\mu\nu}k_\nu$. Feynman rules for other vertices, such as four-photon vertex and ghost vertex, are independent of AMM of fermion. So that we ignore them in this letter.

III. FERMION VERTEX FUNCTION IN NCQED

Up to one-loop level, there are two one-particle irreducible diagrams which contribute to fermion vertex function (fig. 1-(a) and fig. 1-(b)).

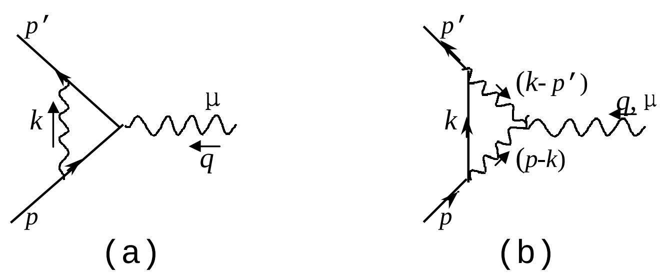


FIG. 1. One-loop correction to fermion vertex function in NCQED

Direct calculation shows that contribution from fig. 1-(a) is same to usual commutative QED, i.e., there is no nonplanar diagrams in fig. 1-(a). However, the fig. 2-(a) entirely new. It denotes contribution from noncommutativity at one-loop level, and corrects fermion vertex function of usual QED. Since there is additional tensor $\theta^{\mu\nu}$, general Lorentz structure of vertex function is rather complicated in NCQED. Concretely, the formal structure of fermion vertex function $\delta\Gamma^\mu$, which is yielded by fig. 1-(b), is follow

$$\begin{aligned} e^{\frac{i}{2}p\times_\theta p'}\delta\Gamma^\mu &= H_1\gamma^\mu+\frac{i}{2m}H_2\sigma^{\mu\nu}q_\nu+\frac{i}{2m}H_3q^\mu+imH_4\tilde{q}^\mu \\ &+H_5(p+p')^\mu\tilde{q}\cdot\gamma+H_6\tilde{q}^\mu\tilde{q}\cdot\gamma, \end{aligned} \tag{7}$$

where $\tilde{q}^\mu=q_\sigma\theta^{\sigma\mu}$, and form factors $H_i(i=1,...,6)$ are function of Lorentz scalars q^2, $q\circ q\equiv g_{\rho\sigma}q_\mu\theta^{\mu\rho}\theta^{\sigma\nu}q_\nu$ [1] and $p\times_\theta p'$. Let us see which form factors will contribute to fermion magnetic moment (we can conveniently take fermion

[1]Our definition on $q\circ q$ has an opposite sign comparing with one in some literatures, since here we work in Minkowski space and use metric $(+,\ -,\ -,\ -)$.

rest framework, such that $p \times_\theta p' = 0$ for $\theta^{0i} = 0$):

1. Conservation of electromagnetic current requires $H_3(q^2 = 0, q \circ q) = 0$. Direct calculation will also shows this point.

2. The form factor $H_1(q^2, q \circ q)$ relates to normalization of electronic charge. In other word, if general vertex function is

$$e^{\frac{i}{2}p \times_\theta p'} \delta\Gamma^\mu = F_1 \gamma^\mu + \text{other independent terms},$$

   normalization of electronic charge requires $F_1(q^2 = 0, q \circ q) = 1$, where $F_1(q^2, q \circ q)$ sum all Feynman diagram contribution and H_1 is included in F_1.

3. Due to $q_\sigma \theta^{\sigma\mu} A_\mu \to \vec{\theta} \cdot \vec{B}$ for $\theta^{0i} = 0$, the H_4 term is independent of fermion spin in nonrelativistic limit. This term denotes a magnetic mass from effect of noncommutativity.

4. The H_6 term can be removed by means of field redefinition

$$A_\mu \longrightarrow A'_\mu = A_\mu + \frac{H_6}{2F_1} \theta^{\rho\sigma} \theta_{\mu\alpha} D^\alpha F_{\rho\sigma}. \tag{8}$$

5. Using Dirac equation for external line fermion, we can obtain

$$\frac{1}{2}(p' - p)_\nu \tilde{q}_\rho \bar{u}(p')[\gamma^\mu, \gamma^\nu]\gamma^\rho u(p) = (p + p')^\mu \bar{u}(p')\tilde{q} \cdot \gamma u(p) + 4(p \times_\theta p')\bar{u}(p')\gamma^\mu u(p). \tag{9}$$

   Then we have

$$(p + p')^\mu \bar{u}(p')\tilde{q} \cdot \gamma u(p) \Longrightarrow i\epsilon^{\mu\nu\sigma\rho} q_\nu \tilde{q}_\sigma \bar{u}(p')\gamma_\rho \gamma_5 u(p). \tag{10}$$

   It is easy to check that $\bar{u}(p')\gamma_i\gamma_5 u(p)$ $(i = 1, 2, 3)$ is independent of fermion spin operator at nonrelativistic limit, i.e., the indices μ, ν, σ in right side of eq. (10) should be spatial indices. Thus for very slowly varing external vector potential, $A_\mu^{\rm cl}(x) = (0, \vec{\mathbf{A}}^{\rm cl}(\vec{\mathbf{x}}))$, we have

$$(p + p')^\mu A_\mu \bar{u}(p')\tilde{q} \cdot \gamma u(p) \quad \propto \quad \epsilon^{jl} \epsilon_{ijk} q_i q_l A_k \quad \propto \quad \sum_k (\partial^2 A_k - \partial_i \partial_k A^i) = 0, \tag{11}$$

   due to equation of motion of external electromagnetic field. Therefore, the H_5 term also does not contribute to fermion magnetic moment.

According to the above analysis, we can conclude that noncommutativity correction to fermion AMM is

$$\delta a_f = H_2(q^2 = 0, q \circ q). \tag{12}$$

Applying the Feynman rules in sect. 2, we find, to order α and for small $m^2 q \circ q$,

$$\begin{aligned}
H_1(q^2 = 0, q \circ q) &= -\frac{3\alpha}{8\pi}[\frac{2}{\epsilon} + \ln{(\mu^2 q \circ q)}] + \mathcal{O}(1), \\
H_2(q^2 = 0, q \circ q) &= -\frac{\alpha}{96\pi} m^2 q \circ q [\ln \frac{m^2 q \circ q}{4} + 2\gamma_{\rm E} - \frac{19}{6}] + \mathcal{O}((m^2 q \circ q)^2), \\
H_4(q^2 = 0, q \circ q) &= -\frac{\alpha}{24\pi} [\ln \frac{m^2 q \circ q}{4} + 2\gamma_{\rm E} - \frac{8}{3}] + \mathcal{O}((m^2 q \circ q)^2),
\end{aligned} \tag{13}$$

where μ is scale factor of dimensional regularization, $\gamma_{\rm E} \simeq 0.5772$ is Euler constant. The UV divergence for $\epsilon \to 0$ in H_1 is from planar diagram, and IR divergence for $q \circ q \to 0$ (precisely it should be called "UV/IR mixing") in H_1 and H_4 are from nonplanar diagram. Fortunately, there is neither UV nor IR divergence in H_2. Thus noncommutativity correction to fermion AMM is finite at one-loop level. In addition, direct evaluations show that there are no further UV divergence in H_5 and H_6, thus NCQED are also renormalizable up to one-loop.

IV. NONCOMMUTATIVITY CORRECTION TO MUON AMM

Now let us return to muon AMM in E821 experiment. It is convenient to take the homogeneous magnetic field is along x^3 direction. Then noncommutative parameters $\theta^{\mu\nu}$ are given by

$$\theta^{0i} = 0, \qquad \theta^{13} = \theta^{23} = 0, \qquad \theta^{12} = -\theta^{21} = \theta = \frac{\hbar c}{eB}. \tag{14}$$

In particular, in muon rest framework the parameter θ is dilated to $\theta = \hbar c/(eB\gamma_{\mathrm{L}}^2) \simeq 5.3 \times 10^{-15}\mathrm{cm}^2 \simeq 1.36 \times 10^7\mathrm{MeV}^{-2}$, where magnetic induction $B \simeq 14.5K\mathrm{G}$, and $\gamma_{\mathrm{L}} \simeq 29.3$ is Lorentz factor.

In microscopic description of interaction between external magnetic field and muon, the dominant effect is that very low energy photon is scattered by relativistic muon (inverse Compton scattering), and photon obtain higher energy after scattering (synchronous radiation in classical electrodynamics). Since muon always loses energy in this mechanicsm, it stays at the lowest Landau level in the most time. In addition, the inverse Compton scattering tell us $q_3 \simeq 0$ in muon rest framework (where $q_\mu = (E_\gamma, \vec{q})$ denotes four-moment of incident photon). Then we have $q \circ q \simeq \theta^2 E_\gamma^2$. It is impossible to exactly evaluate numeric result of noncommutativity correcion to muon AMM. The reason is that we can not know E_γ exactly, or rather, E_γ distributes in large region. In this sense, the noncommutativity correcion to muon AMM is statistical. From inverse Compton scattering we have $E_\gamma \sim E_\gamma^{^{SR}}/\gamma_{\mathrm{L}}^2$, where $E_\gamma^{^{SR}}$ is photon energy in synchronous radiation). In addition, the spectrum distribution function of synchronous radiation in $x^1 - x^2$ plane is well-known

$$N_{\Delta\omega}(\omega^{^{SR}}) \propto \omega^{^{\mathrm{SR}}} K_{2/3}^2(y/2), \qquad y = \frac{\omega^{^{\mathrm{SR}}}}{\omega_c}, \tag{15}$$

where $K_{2/3}$ is the modified Bessel function of the second kind, ω_c is critical frequency of synchronous radiation,

$$\omega_c = \frac{3eB}{2m_\mu c}\gamma_{\mathrm{L}}^2 \Longrightarrow (E_\gamma^{^{SR}})_c = \frac{3\hbar eB}{2m_\mu c}\gamma_{\mathrm{L}}^2. \tag{16}$$

The above equation yields that critical energy of incident photon is $E_\gamma^c \sim 3\hbar eB/(2m_\mu c) \simeq 1.2 \times 10^{-12}\mathrm{MeV}$, which is indeed very small. Then using eq. (13), we obtain the noncommutativity correction to muon AMM as follow

$$\begin{aligned}\delta a_\mu &\simeq -\frac{\alpha}{96\pi}\frac{m_\mu^2\theta^2(E_\gamma^{^{SR}})_c^2}{\gamma_{\mathrm{L}}^4}\frac{\int_0^\infty y^3\{\ln\frac{m_\mu^2\theta^2(E_\gamma^{^{SR}})_c^2}{4\gamma_{\mathrm{L}}^4} + 2\gamma_{\mathrm{E}} - \frac{19}{6} + \ln y^2\}K_{2/3}^2(y/2)dy}{\int_0^\infty yK_{2/3}^2(y/2)dy} \\ &= -\frac{9\alpha}{384\pi\gamma_{\mathrm{L}}^4}\frac{\int_0^\infty y^3\{\ln\frac{9}{16\gamma_{\mathrm{L}}^4} + 2\gamma_{\mathrm{E}} - \frac{19}{6} + \ln y^2\}K_{2/3}^2(y/2)dy}{\int_0^\infty yK_{2/3}^2(y/2)dy} \\ &= 1.57 \times 10^{-9}.\end{aligned} \tag{17}$$

It is suprise that noncommutativity correction makes SM prediction close to experiment, and also is order to 10^{-9}.

We shall conclude with several remarks. First, our study in this letter is an attempt rather than an affirmable conclusion. The quantum mechanics states that, when a charge particle is in the lowest landau level, coordinates of cneter of its wave package (not any spatial coorinates) do not commute. Meanwhile, the noncommutative field theory used by this letter has a prior assumption that arbitrary coordinates in different directions fail to commute. Therefore, the noncommutative field theory only is an approximate description on single charge particle lying the lowest Landau level. We still do not know how to evaluate error bar of the approximation. Of course, the higher energy of particle (or the shorter its Compton wave length) is, the better this approximation is. In addition, it has been expected that noncommutativity is an intrinsic propertiy rather than induction of proper background field. This effect, however, will correct muon AMM in very tiny order of magnitude. For example, if we assume that length of noncommutativity is smaller than classical electron radius, i.e., $\theta < 10^{-30}\mathrm{cm}^2 = 10^{-8}\mathrm{MeV}^{-2}$, the correction to muon AMM will be smaller than 10^{-37}.

Secondly, from the second line of eq. (17), we can see that noncommutativity correction to fermion AMM is independent of mass of fermion in relativistic limit. It implies that, if configuration propsed by this letter is right, to measure electron AMM in environment of E821 experiment, we will find same correction on electron AMM. This can also test whether noncommutative field theory is a good approximation to describe single fermion lying the lowest Landau level.

There are three and four photon vertices in NCQED. However, we should remember that, in our consideration the noncommutativity locates in coordinates of position of fermion. So that three and four photon vertices are virtual, and only exist in fermionic interaction. It is impossible that all photon are on-shell in these vertices.

Finally we note that there are some theoretical problems on NCQED needed to be solved: direct renormalization calculation and β function of NCQED; infrared safety for limit $q \circ q \to 0$, etc. These problems will be studied in forthcoming papers.

We thank professors Y.-L. Wu and M. Li for discussions which clarifing some physical understanding of our results.

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